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SUBMERSIONS FROM ANTI-DE SITTER SPACE WITH TOTALLY GEODESIC FIBERS

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Introduction

In [5] O'Neill introduced the notion of a Riemannian submersion. Escobales [1], [2] classified Riemannian submersions from a sphere S^n and from a complex projective space $\mathbf{C}P^n$ with totally geodesic fibers.

This paper investigates such submersions for an indefinite space form: anti-de Sitter space. It is shown that there is essentially only one submersion from H_1^{2n+1} onto a Riemannian manifold with totally geodesic fibers, and this is the standard one onto a complex hyperbolic space $\mathbf{C}H^n$.

1. Let M, B be C^∞ indefinite Riemannian manifolds. An indefinite Riemannian submersion π: $M \to B$ is an onto, C^∞ mapping such that

(1) π is of maximal rank,

(2) π_* preserves the lengths of horizontal vectors, i.e., vectors orthogonal to the fibers $\pi^{-1}(x)$, $x \in B$,

(3) the restriction of the metric to the vertical vectors is nondegenerate.

Consider the following example, [4, p. 282, Example 10.7] p: $H_1^{2n+1} \to \mathbf{C}H^n$, where H_1^{2n+1} is a $(2n+1)$-dimensional anti-de Sitter space with constant sectional curvature -1 and signature $(1, 2n)$, and $\mathbf{C}H^n$, defined below, is a complex hyperbolic space. On $\mathbf{C}^{n+1}$ let

$$(\vec{z}, \vec{w}) = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k,$$

$$\langle \vec{z}, \vec{w} \rangle = Re(\vec{z}, \vec{w}) = -x_0u_0 - y_0v_0 + \sum_{k=1}^{n} x_ku_k + y_kv_k,$$

where

$$\vec{z} = (z_0, \cdots, z_n) = (x_0 + iy_0, \cdots, x_n + iy_n),$$
$$\vec{w} = (w_0, \cdots, w_n) = (u_0 + iv_0, \cdots, u_n + iv_n),$$
$$H_1^{2n+1} = \{\vec{z} \in \mathbf{C}^{n+1}: (\vec{z}, \vec{z}) = -1 = \langle \vec{z}, \vec{z} \rangle\}$$
$$= \{(x_0, y_0, \cdots, x_n, y_n): -x_0^2 - y_0^2 + x_1^2 + \cdots + x_n^2 + y_n^2 = -1\}.$$

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The tangent space to H_1^{2n+1} at $\vec{z}$, $T_{\vec{z}}$ is

$$T_{\vec{z}} = \{ W \in \mathbf{C}^{n+1} \colon \langle \vec{z}, W \rangle = 0 \}.$$

Let $T'_{\vec{z}} = \{ U \in \mathbf{C}^{n+1} \colon \langle U, \vec{z} \rangle = 0 = \langle U, i\vec{z} \rangle \}$, and setting $H_1^1 = \{ \lambda \in \mathbf{C} \colon \lambda\bar{\lambda} = 1 \}$ we have an H_1^1 action on H_1^{2n+1}, $\vec{z} \mapsto \lambda\vec{z}$.

At each point of H_1^{2n+1} the vector field $i\vec{z}$ is tangent to the flow of the action, and $\langle i\vec{z}, i\vec{z} \rangle = -1$. Note that the orbit is $x_t = (\cos t + i \sin t)\vec{z}$ and $dx_t/dt = ix_t$. The orbit lies in the negative definite plane spanned by $\{\vec{z}, i\vec{z}\}$. The identification space of this action is called $\mathbf{C}H^n$, and the projection is denoted by p. It is easy to see that $T_{p(z)}(\mathbf{C}H^n)$ can be identified with $T'_{\vec{z}}$. This construction mimics that of $\mathbf{C}P^n$. $\mathbf{C}H^n$ has negative constant holomorphic sectional curvature. $p \colon H_1^{2n+1} \to \mathbf{C}H^n$ is an indefinite Riemannian submersion.

The main result of this paper is

Theorem 1. *If $\pi \colon H_1^k \to B^j$ is an indefinite Riemannian submersion from anti-de Sitter space to a Riemannian manifold with totally geodesic fibers, then $k = 2n + 1$, $j = 2n$, and B^{2n} is holomorphically isometric to $\mathbf{C}H^n$, where B^j is equipped with an integrable almost complex structure induced from the submersion.* (*See* [1], [2].)

2. This section deals with the algebraic preliminaries.

Given $\pi \colon M \to B$, an indefinite Riemannian submersion, let V and H denote the vertical and horizontal projections.

$$T_x(M) = V_x \otimes H_x$$
$$V \swarrow \qquad \searrow H$$
$$V_x \qquad\qquad H_x$$

O'Neill [5] defines two fundamental tensors on $(M, \nabla, \langle\ ,\ \rangle)$:

$$A_E F = V(\nabla_{HE} HF) + H(\nabla_{HE} VF), \qquad T_E F = H(\nabla_{VE} VF) + V(\nabla_{VE} HF),$$

for vector fields E, F on M. These two tensors have the following properties:

(i) $A_{HE} = A_E$; $T_{VE} = T_E$.

(ii) A_E and T_E are skew-symmetric with respect to $\langle\ ,\ \rangle$.

(iii) A_E and T_E take vertical vectors to horizontal vectors and vice-versa.

(iv) If V and W are vertical and X and Y are horizontal, then

$$T_V W = T_W V, \quad A_Y X = -A_X Y.$$

Definition. A vector field X on M is said to be *basic* if it is the unique horizontal lift of a vector field X_* on B, so that $\pi_*(X) = X_*$.

Lemma 1 [5, *p.* 460]. *If X and Y are basic vector fields on M, then*
(1) $\langle X, Y\rangle = \langle X_*, Y_*\rangle \cdot \pi$;
(2) *$H[X, Y]$ is the basic vector field corresponding to $[X_*, Y_*]$,*
(3) *$H(\nabla_X Y)$ is the basic vector field corresponding to $\nabla^*_{X_*} Y_*$ where ∇^* is the connection on B.*

Lemma 2 [5, *p.* 461]. *If ∇ is the connection on M, and $\hat{\nabla}$ the connection on a fiber, then for X, Y horizontal vector fields and V, W vertical vector fields we have*
(1) $\nabla_V W = T_V W + \hat{\nabla}_V W$,
(2) $\nabla_V X = H(\nabla_V X) + T_V X$,
(3) $\nabla_X V = A_X V + V(\nabla_X V)$,
(4) $\nabla_X Y = H(\nabla_X Y) + A_X Y$,
(5) *if X is basic, then $H(\nabla_V X) = A_X V$.*

We will assume that the fibers are totally geodesic, so that by (1) $T_V W = 0$, which gives
(1)′ $\nabla_V W = \hat{\nabla}_V W$,
(2)′ $\nabla_X V = H(\nabla_V X)$.

O'Neill also proves [5, p. 465] the following relations between the sectional curvatures K of M and K_* of B when the fibers are totally geodesic:

$$(\theta)\qquad K_{X\wedge V} = \frac{\langle A_X V, A_X V\rangle}{\langle X, X\rangle\langle V, V\rangle},$$

$$(\theta\theta)\qquad K_{*X_*\wedge Y_*} = K_{X\wedge Y} + \frac{3\langle A_X Y, A_X Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2},$$

where X and Y are horizontal vector fields, V is a vertical vector field, and $K_{E\wedge F}$ (respectively, $K_{*E_*\wedge F_*}$) denotes the sectional curvature in M (respectively B) of the plane spanned by E and F (E_* and F_*).

In the Riemannian case, $(\theta\theta)$ says that sectional curvatures are increased by submersions. Since we will be dealing with submersions from H_1^{m+k}, let us first look at the case of submersion from a Lorentzian manifold with negative sectional curvature to a Riemannian manifold.

Proposition 1. *If π: $M_1^{m+k} \to B^m$ is an indefinite Riemannian submersion with totally geodesic fibers, where M is Lorentzian and has negative sectional curvature and B is Riemannian, then $k = 1$.*

Proof. By (θ) we have

$$0 > K_{X\wedge V} = \frac{\langle A_X V, A_X V\rangle}{\langle X, X\rangle\langle V, V\rangle}.$$

Since $A_X V$ and X are horizontal, $\langle A_X V, A_X V\rangle \geqslant 0$ and $\langle X, X\rangle > 0$. Thus $\langle V, V\rangle < 0$, i.e., V is timelike, and $A_X V \neq 0$ for all horizontal $X \neq 0$, and all vertical $V \neq 0$. Since M is Lorentzian, the timelike vectors are essentially one-dimensional and so the vertical vectors are one-dimensional. q.e.d.

Thus if $\pi: H_1^{m+1} \to B^m$ is a submersion with totally geodesic fibers, then by $(\theta\theta)$ we have

$$K_{*X_* \wedge Y_*} = -1 + \frac{3\langle A_X Y, A_X Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2},$$

and because $A_X Y$ is vertical, $\langle A_X Y, A_X Y\rangle \leqslant 0$. This shows that $K_* \leqslant -1$ so that curvature is nonincreasing in a submersion of this type.

Proposition 2. *If $\pi: H_1^{m+1} \to B^m$ is a submersion with totally geodesic fibers, then $\pi_j(B^m) = 0, j = 1, 2, 3, \cdots$.*

Hint of proof. We must only show that in the fibration

$$\begin{array}{c} S^1 \xrightarrow{i} S^1 \times \mathbf{R}^m \to B^m \\ \wr\!\wr \\ H_1^{m+1} \end{array}$$

that i induces a homotopy equivalence. This is clear, since every geodesic in H_1^{m+1} is a circle in $\mathbf{R}_2^{m+2}$ of the form $(\cos t)x_0 + (\sin t)X_0$ with $\langle x_0, X_0\rangle = 0$.

Theorem 2. *If $\pi: H_1^{m+1} \to B^m$ is an indefinite Riemannian submersion with totally geodesic fibers, then $m = 2n$, for some $n > 0$.*

Proof. H_1^{m+1} is not only equipped with the fundamental tensor A but also with a foliation by timelike geodesics. Thus there is a smooth vector field V tangent to these geodesics with $\langle V, V\rangle = -1$. Let X and Y be horizontal vector fields on H_1^{m+1}. We know that $A_X V$ is horizontal. Therefore

$$0 = Y\langle X, V\rangle = \langle \nabla_Y X, V\rangle + \langle X, \nabla_Y V\rangle = \langle A_Y X, V\rangle + \langle X, A_Y V\rangle.$$

Interchanging X and Y we have

$$0 = \langle A_X Y, V\rangle + \langle Y, A_X V\rangle.$$

Since $A_X Y + A_Y X = 0$, adding these two equations yields

$$\langle X, A_Y V\rangle + \langle Y, A_X V\rangle = 0,$$

so that $A\_V: H_x \to H_x$ is skew-symmetric. If the horizontal space H_x were odd dimensional, then A_V would have 0 as an eigenvalue. On the other hand, (θ) gives

$$\frac{\langle A_X V, A_X V\rangle}{\langle X, X\rangle\langle V, V\rangle} = -1.$$

But $\langle V, V\rangle = -1$, so $\langle A_X V, A_X V\rangle = \langle X, X\rangle$ which means A_V is an isometry. Thus H_x must be even dimensional, and $m = 2n$. q.e.d.

In fact a skew-symmetric isometry is an almost complex structure, since a basis can be found with respect to which the mapping is of the form

$$\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}.$$

Thus we know that any indefinite Riemannian submersion from H_1^k with totally geodesic fibers onto a Riemannian manifold is of the form $\pi\colon H_1^{2n+1} \to B^{2n}$, and B^{2n} is simply connected.

3. This part of the paper will show that B^{2n} is holomorphically isometric to D^n, the disc in $\mathbf{C}^n$ with the Bergman metric [4, Ex. 10.7].

First we shall show that the submersion induces an almost complex structure on B^{2n} and a Hermitian metric on B^{2n}. Then it will be seen that with these induced structures B^{2n} is a Kähler manifold.

One could also show that H_1^{2n+1} is an indefinite regular Sasakian manifold with the structure induced from the submersion and so [6, p. 150] B^{2n} is a real $2n$-dimensional Kähler manifold. The proofs are similar.

Let V be as in the proof of Theorem 2. Since V is a geodesic vector field, $\nabla_V V = 0$. Let $\phi(E) = A_E V$ for all vector fields E on H_1^{2n+1}, and let η be the one-form dual to V, so that $\eta(V) = -1$. Then we have

Lemma 3. (1) $\phi(V) = 0$,
(2) $\eta(\phi(E)) = 0$,
(3) $\phi^2(E) = -E - \eta(E)V$,
(4) $\langle\phi(E), \phi(F)\rangle = \langle E, F\rangle + \eta(E)\eta(F)$,
(5) $\eta(E) = \langle E, V\rangle$,
for all vector fields E, F on H_1^{2n+1}.

Proof. (1), (2), (5) are clear.

(3) Let $E = X + \lambda V$ where X is horizontal. Then

$$\phi^2(E) = A_{A_E V} V = A_{A_X V} V, \text{ and } A_{A_X V} V = -X,$$

since for all horizontal Y

$$\begin{aligned}\langle A_{A_X V} V, Y\rangle &= -\langle V, A_{A_X V} Y\rangle = \langle V, A_Y A_X V\rangle \\ &= -\langle A_Y V, A_X V\rangle = -\langle X, Y\rangle.\end{aligned}$$

Thus

$$\phi^2(X + \lambda V) = -X = -(X + \lambda V) - \eta(X + \lambda V)V = -E - \eta(E)V.$$

(4) Let $E = X + \lambda V$, $F = Y + \mu V$ where X and Y are horizontal. Then

$$\begin{aligned}\langle \phi E, \phi F\rangle &= \langle A_E V, A_F V\rangle = \langle A_X V, A_Y V\rangle \\ &= \langle X, Y\rangle = \langle X + \lambda V, Y + \mu V\rangle + \eta(X + \lambda V)\eta(Y + \mu V).\end{aligned}$$

q.e.d.

Since the basic vector fields on H_1^{2n+1} correspond to vector fields on B^{2n}, we focus our attention on these vector fields. In particular, in order to have ϕ induce an almost complex structure on B^{2n}, if X is basic, then $A_X V$ must be basic.

Theorem 3. *If X is a basic vector field on H_1^{2n+1}, then $A_X V$ is a basic vector field.*

Proof. Lemma 1.2 [1, p. 254]: Let B_i be a basic vector field on H_1^{2n+1} corresponding to B_{i_*} on B^{2n}, and let X be horizontal. If $\langle X, B_i\rangle_p = \langle X, B_i\rangle_{p'}$ for all such B_i and any p, p' in $\pi^{-1}(b)$, $b \in B^{2n}$, then X is basic.

This means that for all B, basic, we must show that $V\langle A_X V, B\rangle = 0$. Since

$$\begin{aligned} V\langle A_X V, B\rangle &= \langle \nabla_V(A_X V), B\rangle + \langle A_X V, \nabla_V B\rangle \\ &= \langle \nabla_V(A_X V), B\rangle + \langle A_X V, A_B V\rangle \\ &= \langle \nabla_V(A_X V), B\rangle + \langle X, B\rangle, \end{aligned}$$

we must show that for X basic $\nabla_V(A_X V) = -X$. On H_1^{2n+1}

$$R(V, X)V = \nabla_V\nabla_X V - \nabla_X\nabla_V V - \nabla_{[X, V]}V = -(V \wedge X)V,$$

since H_1^{2n+1} has constant curvature -1.

$R(V, X)V = \nabla_V\nabla_X V - \nabla_{[X, V]}V$ since $\nabla_V V = 0$, and because $[V, X]$ is vertical $\nabla_{[X, V]}V = \rho\nabla_V V = 0$ yielding $R(V, X)V = \nabla_V\nabla_X V$.

On the other hand

$$R(V, X)V = -(\langle X, V\rangle V - \langle V, V\rangle X) = -X$$

so $\nabla_V\nabla_X V = -X$. But

$$\nabla_V(\nabla_X V) = \nabla_V(A_X V + V(\nabla_X V)) = \nabla_V(A_X V)$$

since $\langle \nabla_X V, V\rangle = \frac{1}{2}X\langle V, V\rangle = 0$. q.e.d.

Thus ϕ induces an almost complex structure on B^{2n}.

Theorem 4. *This almost complex structure on B^{2n} is integrable.*

Proof. We must show that $N_\phi(X_*, Y_*) = 0$ where X_* and Y_* are vector fields on B^{2n}, and N_ϕ is the Nijenhuis tensor of ϕ:

$$N_\phi(X_*, Y_*) = [\phi X_*, \phi Y_*] - [X_*, Y_*] - \phi[X_*, \phi Y_*] - \phi[\phi X_*, Y_*].$$

The basic vector field corresponding to $N_\phi(X_*, Y_*)$ is $H[\phi X, \phi Y] - H[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y]$ where X and Y are the basic vector fields associated with X_* and Y_*. This is equivalent to

$$H(\nabla_{\phi X}\phi Y) - H(\nabla_{\phi Y}\phi X) - H(\nabla_X Y) + H(\nabla_Y X) - \phi(\nabla_X \phi Y) + \phi(\nabla_{\phi Y} X) - \phi(\nabla_{\phi X} Y) + \phi(\nabla_Y \phi X)$$

$$= \overset{(a)}{H\big(\nabla_{(A_X V)}(A_Y V)\big)} - \overset{(b)}{H\big(\nabla_{(A_Y V)}(A_X V)\big)} - \overset{(c)}{H(\nabla_X Y)} + \overset{(d)}{H(\nabla_Y X)}$$

$$= \overset{(e)}{A_{\nabla_X (A_Y V)} V} + \overset{(f)}{A_{\nabla_{(A_Y V)} X} V} - \overset{(g)}{A_{\nabla_{(A_X V)} Y} V} + \overset{(h)}{A_{\nabla_Y (A_X V)} V}.$$

In order to prove $N_\phi(X_*, Y_*) = 0$ it is sufficient to prove

Lemma 4. *If X and Y are horizontal vector fields on H_1^{2n+1}, then*

$$(\dagger) \qquad H(\nabla_X (A_Y V)) = A_{(\nabla_X Y)} V.$$

If (†) holds, then

$$H\big(\nabla_{A_X V} A_Y V\big) = A_{\nabla_{A_X V} Y} V,$$
$$H\big(\nabla_{A_Y V} A_X V\big) = A_{\nabla_{A_Y V} X} V,$$
$$A_{\nabla_X (A_Y V)} V = H\big(\nabla_X \big(A_{A_Y V} V\big)\big) = -H(\nabla_X Y),$$
$$A_{\nabla_Y (A_X V)} V = -H(\nabla_Y X),$$

and so (a) = (g), (b) = (f), (e) = –(c) and (h) = –(d). Thus the sum is zero.

Proof of Lemma 4. (†) is equivalent to

$$(\dagger') \qquad \langle \nabla_X A_Y V, Z\rangle = \langle A_{\nabla_X Y} V, Z\rangle \text{ for all horizontal } Z.$$

From [5, p. 464 {3}]

$$\langle R(Y, Z)X, V\rangle = -\langle (\nabla_X A)_Y Z, V\rangle,$$

so

$$\langle R(Y, Z)V, X\rangle = \langle (\nabla_X A)_Y Z, V\rangle.$$

Since $R(Y, Z)V = -(Y \wedge Z)V = 0$, we have $\langle (\nabla_X A)_Y Z, V\rangle = 0$, which expands to

$$0 = \langle \nabla_X (A_Y Z), V\rangle - \langle A_{\nabla_X Y} Z, V\rangle - \langle A_Y (\nabla_X Z), V\rangle.$$

Substituting

$$A_Y Z = -\langle A_Y Z, V\rangle V = \langle A_Y V, Z\rangle V$$

in the above equation gives

$$\begin{aligned} 0 &= \langle \nabla_X \langle A_Y V, Z\rangle V, V\rangle - \langle A_{\nabla_X Y} Z, V\rangle - \langle A_Y(\nabla_X Z), V\rangle \\ &= \langle A_Y V, Z\rangle \langle \nabla_X V, V\rangle + \langle X\langle A_Y V, Z\rangle V, V\rangle \\ &\quad -\langle A_{\nabla_X Y} Z, V\rangle - \langle A_Y(\nabla_X Z), V\rangle \\ &= -\langle \nabla_X(A_Y V), Z\rangle - \langle A_Y V, \nabla_X Z\rangle - \langle A_{\nabla_X Y} Z, V\rangle - \langle A_Y(\nabla_X Z), V\rangle \\ &= \langle \nabla_X(A_Y V), Z\rangle + \langle A_Y V, \nabla_X Z\rangle - \langle Z, A_{\nabla_X Y} V\rangle + \langle A_Y(\nabla_X Z), V\rangle \\ &= \langle \nabla_X(A_Y V), Z\rangle - \langle A_{\nabla_X Y} V, Z\rangle \end{aligned}$$

because $\langle A_Y V, \nabla_X Z\rangle + \langle A_Y(\nabla_X Z), V\rangle = 0$. q.e.d.

Note that the metric induced on B^{2n} is Hermitian since $\langle \phi X, \phi Y\rangle = \langle X, Y\rangle$ for X, Y basic on H_1^{2n+1}. Thus in order to show that B^{2n} is Kählerian we must only show that

$$\nabla^*_{X_*}\phi Y_* = \phi\left(\nabla^*_{X_*} Y_*\right).$$

Since the basic vector field corresponding to $\nabla^*_{X_*} Y_*$ is $H(\nabla_X Y)$ and the basic vector field corresponding to $\nabla^*_{X_*}\phi Y_*$ is $H(\nabla_X \phi Y)$, we must show that

$$H(\nabla_X \phi Y) = \phi(\nabla_X Y)$$

for X, Y basic on H_1^{2n+1}. But this is just (†).

Thus B^{2n} is a Kähler manifold, $\pi_1(B^{2n}) = 0$ and to finish the proof of Theorem 1 it is only necessary to show that B^{2n} has constant holomorphic sectional curvature [4, p. 170, Theorem 7.9].

By $(\theta\theta)$ we obtain

$$\begin{aligned} K_{*X_* \wedge \phi X_*} &= K_{X\wedge \phi X} + 3\frac{\langle A_X \phi X, A_X \phi X\rangle}{\langle X, X\rangle\langle \phi X, \phi X\rangle - \langle X, \phi X\rangle^2} \\ &= -1 + 3\frac{\langle A_X A_X V, A_X A_X V\rangle}{\langle X, X\rangle^2}. \end{aligned}$$

Note $A_X A_X V = -\langle A_X A_X V, V\rangle V = \langle A_X V, A_X V\rangle V = \langle X, X\rangle V$, so that

$$K_{*X_* \wedge \phi X_*} = -1 + 3\frac{\langle X, X\rangle^2 \langle V, V\rangle}{\langle X, X\rangle^2} = -4.$$

This completes the proof of Theorem 1.

Just as Escobales does in [1] we can show that any two such maps are equivalent.

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